Quantum corrections to the motion of classical charges in high intensity electromagnetic fields

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The motion of classical test particles in the nonlinear electrodynamics described by the Weisskopf effective Lagrangian is studied. A technique using the introduction of an extra coordinate is exploited to obtain the equation of motion for a test particle in the nonlinear electromagnetic field. The method has the advantage that the observed mass and charge appear directly in the equation of motion. Quantum corrections due to vacuum polarization effects are given for the Lorentz force law. [S1063-651X(96)07007-9]

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I. INTRODUCTION

One of the few nonperturbative results of quantum electrodynamics is the effective electromagnetic field Lagrangian first derived by Weisskopf and Dan [1] who followed the earlier work of Euler, Kockel, and Heisenberg [2–4]. The Weisskopf Lagrangian was later rederived in a different way by Schwinger [5] using the proper time formalism. This Lagrangian utilizes an analytic solution of the Dirac equation for an electron in an external, slowly varying field. The polarization of the vacuum due to electron pairs described by this solution is fully taken into account to all orders of the external field, but excludes radiative corrections due to virtual photons. It is a nonperturbative result as far as the external field is concerned.

The Weisskopf effective Lagrangian is a functional of the two field invariants

$$\mathcal{F}=\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}=\tfrac{1}{2}(\vec{B}^2-\vec{E}^2) \tag{1.1}$$

and

$$\mathcal{G}=\tfrac{1}{4}F^*_{\mu\nu}F^{\mu\nu}=-\tfrac{1}{2}(\vec{E}\cdot\vec{B}), \tag{1.2}$$

where the star denotes the dual $^*F^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. It does not depend on the derivatives of these invariants which are assumed to be negligible [6]. Using the complex quantity $\mathcal{X}$ defined by

$$\mathcal{X}^2=(\vec{B}+i\vec{E})^2=2(\mathcal{F}-i\mathcal{G}), \tag{1.3}$$

Schwinger [5] has written this Lagrangian in the form

$$\mathcal{L}=-\mathcal{F}-\frac{1}{8}\pi^{-2}\int_0^\infty s^{-3}\exp(-m^2s)\times\left[(es)^2\mathcal{G}\frac{\operatorname{Re}\cosh(es\mathcal{X})}{\operatorname{Im}\cosh(es\mathcal{X})}-1-\frac{2}{3}\mathcal{F}\right]ds. \tag{1.4}$$

The weak field expansion yields the well known result [7–9]

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{2\alpha^2}{45m^4}\left[\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right)^2+7\left(\frac{1}{4}F^*_{\mu\nu}F^{\mu\nu}\right)^2\right]=\frac{1}{2}(\vec{E}^2-\vec{B}^2)+\frac{2\alpha^2}{45m^4}[(\vec{E}^2-\vec{B}^2)^2+7(\vec{E}\cdot\vec{B})^2], \tag{1.5}$$

where $\alpha=e^2/4\pi=e^2/4\pi\approx1/137$ is the fine-structure constant and m is the rest mass of the electron. For a large field the Lagrangian (1.4) is highly nonlinear and is the appropriate point of departure to begin a study of nonlinear effects. For practical fields, however, it is sufficient to use the Lagrangian of (1.5) to investigate the lowest order nonlinear effects arising from quantum corrections.

While the free field equations resulting from these nonlinear Lagrangians have been studied extensively [8], the important problem of the motion of a classical test particle in such a field has apparently never been addressed. In particular, the equation of motion of a test particle in an electromagnetic field described by (1.4) or (1.5) has never been obtained. The purpose of this communication is to derive the effective equation of motion for a test charge in a field described by this nonlinear effective Lagrangian. One expects, of course, to obtain the Lorentz force in such an equation of motion as well as an additional term due to quantum corrections.

The primary difficulty in obtaining the equation of motion is that one must be careful that only the observed charge and mass of the test particle appear in the final result. Clearly since vacuum polarization effects are included in (1.4), to be consistent they should also be taken into account in the equation of motion for a test particle. It is not immediately clear how to do this in a gauge invariant fashion within the usual framework of quantum electrodynamics. In order to avoid these renormalization problems and ensure that only the observed mass and charge enter into the equation of motion, a Kaluza-Klein type formulation is given for the nonlinear electrodynamics. Since the equation of motion for test particles in a Kaluza-Klein theory is just the geodesic equation, one is able to use the formulation to obtain the equation of motion for a test charge in a straightforward method. At the classical level the equation of motion derived in this manner from the Kaluza-Klein formalism is known to be identical to the equation of motion one would obtain from the regular four-space formulation.

The advantage of the Kaluza-Klein approach is that the identification of the observed mass and charge is readily apparent, so the equation of motion can be unambiguously determined. Moreover, it is an interesting problem in its own right of finding a Kaluza-Klein type theory that will yield the Lagrangian of Eq. (1.5). Theories with extra dimensions have a long history [10,11] but have in recent years attracted renewed interest [12]. It is an interesting problem from the standpoint of these theories to see how quantum and nonlinear corrections can be incorporated into a simplified Kaluza-Klein model.

In Sec. II the relevant features of the Kaluza-Klein theory with a single extra dimension are summarized. This is followed in Sec. III by a description of how this can be used to obtain a Kaluza-Klein type formulation for the nonlinear electrodynamics described by the Weisskopf Lagrangian. In Sec. IV this formulation is used to find the equations of motion for a classical test particle in this theory. Section V contains concluding remarks.

II. FLAT KALUZA-KLEIN THEORY

As a preliminary which will also serve to introduce the notation, the pertinent aspects of the Kaluza-Klein theory will be summarized. It is sufficient for the purposes here to consider only the flat (no gravity) Kaluza-Klein type theory given by a metric of the form

$$d\mathcal{S}^2=\eta_{\mu\nu}dx^\mu dx^\nu-f^{-1}(d\theta-A_\nu dx^\nu)^2, \tag{2.1}$$

where $A_\nu(x^\mu)$ is the usual electromagnetic four-vector potential and $f(x^\mu)$ is a scalar field to be specified later. The extra coordinate, θ, is assumed compactified and normalized so that $0\leqslant\theta<2\pi$. It is convenient to introduce the notation $x^0=t,x^1=x,x^2=y,x^3=z,x^5=\theta$ and write (2.1) in the form

$$d\mathcal{S}^2=g_{AB}dx^A dx^B, \tag{2.2}$$

where

$$g_{AB}=\begin{bmatrix}\eta_{\mu\nu}-f^{-1}A_\mu A_\nu & f^{-1}A_\mu\\ f^{-1}A_\nu & -f^{-1}\end{bmatrix}. \tag{2.3}$$

The summation convention is used throughout with upper case latin indices A,B, etc., running over the five dimensions 0,1,2,3,5; lower case Greek indices μ,ν, etc., range over the usual four-space values 0,1,2,3; and lower case latin indices i,j, etc. run over the physical space dimensions 1,2,3. The Minkowski space metric $\eta_{\mu\nu}$ is diagonal with signature $(+1, -1, -1, -1)$ and natural units $\hbar=c=1$ are used.

In the absence of additional fields, the field equations follow from the Kaluza-Klein type action

$$\mathcal{A}_F=-\frac{1}{2}\pi^{-1}\int R\times|\det(g_{AB})|^{1/2}d^5x, \tag{2.4}$$

where R is the five-dimensional curvature scalar resulting from the metric g_{AB} of (2.3). The curvature scalar is most easily calculated using the Cartan formalism. The details of this calculation are outlined below.

One introduces the basis of one-forms

$$\sigma^\mu=dx^\mu, \tag{2.5a}$$

$$\sigma^5=f^{-1/2}(d\theta-A_\nu dx^\nu), \tag{2.5b}$$

so that the metric (2.1) can be written in the form

$$d\mathcal{S}^2=\eta_{AB}\sigma^A\sigma^B, \tag{2.6}$$

with

$$\eta_{AB}=\begin{bmatrix}\eta_{\mu\nu} & 0\\ 0 & -1\end{bmatrix}. \tag{2.7}$$

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From (2.5) one readily calculates

$$d\sigma^\mu=0 \tag{2.8a}$$

$$d\sigma^5=-f^{-1/2}f_{,\mu}\sigma^\mu\wedge\sigma^5-f^{-1/2}F_{\mu\nu}\sigma^\mu\wedge\sigma^\nu, \tag{2.8b}$$

where

$$F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}. \tag{2.9}$$

Now by the Cartan structural equation

$$d\sigma^A=\sigma^B\wedge\Omega^A_B, \tag{2.10}$$

with

$$\Omega_{AB}=\eta_{AC}\Omega^C_B,\quad \Omega_{AB}=-\Omega_{BA} \tag{2.11}$$

one can obtain the connection one-form Ω^A_B by (2.10) and (2.8). Doing this one easily finds

$$\Omega^\mu_\nu=-\tfrac{1}{2}f^{-1/2}F^\mu_\nu\sigma^5, \tag{2.12a}$$

$$\Omega^5_\mu=-\tfrac{1}{2}f^{-1}f_{,\mu}\sigma^5-\tfrac{1}{2}f^{-1/2}F_{\mu\nu}\sigma^\nu. \tag{2.12b}$$

Since Ω^A_B is a one form, it can be expanded on the σ^A basis as

$$\Omega^A_B=\gamma^A_{BC}\sigma^C, \tag{2.13}$$

where γ^A_{BC} are the connection coefficients on the orthonormal basis. Comparing (2.13) and (2.12) allows one to read off

$$\gamma^\mu_{\nu\alpha}=0, \tag{2.14a}$$

$$\gamma^\mu_{\nu5}=\gamma^\mu_{5\nu}=-\tfrac{1}{2}f^{-1/2}F^\mu_\nu, \tag{2.14b}$$

$$\gamma^5_{\mu\nu}=-\tfrac{1}{2}f^{-1/2}F_{\mu\nu}, \tag{2.14c}$$

$$\gamma^5_{\mu5}=-\tfrac{1}{2}f^{-1}f_{,\mu}, \tag{2.14d}$$

$$\gamma^\mu_{55}=-\tfrac{1}{2}f^{-1}f^{,\mu}. \tag{2.14e}$$

The Riemann curvature tensor R^A_{BCD} can then be obtained directly from the curvature form by using

$$d\Omega^A_B+\Omega^{AE}\wedge\Omega_{EB}=\tfrac{1}{2}R^A_{BCD}\sigma^C\wedge\sigma^D. \tag{2.15}$$

Using (2.10), (2.12), and (2.13) in Eq. (2.15) one obtains

$$R^A_{BCD}=\gamma^A_{BD,C}-\gamma^A_{BC,D}+\gamma^A_{BE}\gamma^E_{CD}-\gamma^A_{BE}\gamma^E_{DC}+\gamma^A_{EC}\gamma^E_{BD}-\gamma^A_{ED}\gamma^E_{BC} \tag{2.16}$$

for the Riemann curvature tensor in terms of the connection coefficients on the orthonormal basis. The curvature scalar is then given by

$$R=R^B_B=\eta^{BD}R_{BD}, \tag{2.17}$$

where

$$R_{BD}=R^A_{BAD}. \tag{2.18}$$

From these one uses Eqs. (2.14) to calculate

$$R_{\mu\nu}=\tfrac{1}{2}(f^{-1}f_{,\mu})_{,\nu}-\tfrac{1}{4}f^{-2}f_{,\mu}f_{,\nu}-\tfrac{1}{2}f^{-1}F_{\mu\alpha}F^{\alpha}_{\nu}, \tag{2.19a}$$

$$R_{\mu 5}=-\tfrac{1}{2}(f^{-1/2}F^{\alpha}_{\mu})_{,\alpha}-\tfrac{1}{2}f^{-3/2}F_{\mu\alpha}f^{,\alpha}, \tag{2.19b}$$

$$R_{55}=-\tfrac{1}{2}(f^{-1}f^{,\alpha})_{,\alpha}-\tfrac{1}{2}f^{-2}f^{,\alpha}f_{,\alpha}+\tfrac{1}{4}f^{-1}F_{\alpha\beta}F^{\alpha\beta} \tag{2.19c}$$

and

$$R=(f^{-1}f^{,\alpha})_{,\alpha}-\tfrac{1}{2}f^{-2}f^{,\alpha}f_{,\alpha}+\tfrac{1}{4}f^{-1}F_{\alpha\beta}F^{\alpha\beta}. \tag{2.20}$$

Now the action (2.4) can be written as

$$\mathcal{A}_F=-\frac{1}{2\pi}\int\left(-\frac{3}{2}f^{-2}f^{,\alpha}f_{,\alpha}+f^{-1}f^{,\alpha}{}_{,\alpha}+\frac{1}{4}f^{-1}F_{\alpha\beta}F^{\alpha\beta}\right)f^{-1/2}d^5x, \tag{2.21}$$

where the fact that

$$\det(g_{AB})=f^{-1} \tag{2.22}$$

which follows by direct computation from (2.3) has been used. Simplifying (2.21) one arrives finally at

$$A_F=-\frac{1}{2\pi}\int\left[(f^{-3/2}f^{,\alpha})_{,\alpha}+\frac{1}{2}f^{-3/2}F_{\alpha\beta}F^{\alpha\beta}\right]d^5x, \tag{2.23}$$

$$=-\int\frac{1}{4}\left(\frac{e_0}{r_0}\right)^3 f^{-3/2}F_{\alpha\beta}F^{\alpha\beta}d^4x, \tag{2.24}$$

where the divergence term has been dropped and the integration over θ has been performed assuming compactification with a characteristic charge-mass ratio of e_0/m_0 or, equivalently, a compactified fifth dimension with radius $r_0=e_0^2/m_0$.

This shows that the Kaluza-Klein metric (2.1) is equivalent to the field theory represented by the action (2.24) or its corresponding Lagrangian

$$\mathcal{L}=-\frac{1}{4}\left(\frac{e_0}{r_0}\right)^3 f^{-3/2}F_{\alpha\beta}F^{\alpha\beta}. \tag{2.25}$$

It should be noted that these results are applicable in arbitrary fields and, in particular, do not rely on a weak field or slowly varying field approximation.

It is clear from this Lagrangian that if f is taken to be an independent field variable and varied independently of A_μ then (2.24) with the principle of stationary action leads to singular results. To have a consistent formulation one must then require f to be a functional of A_μ. Futhermore, since one would like to maintain the gauge invariance one expects f to be a functional of the two invariants $\mathcal{F}$ and $\mathcal{G}$, i.e., $f=f(\mathcal{F},\mathcal{G})$.

III. KALUZA-KLEIN FORMULATION FOR THE WEISSKOPF LAGRANGIAN

Using the results of the preceding section it is possible to give a Kaluza-Klein type formulation for the nonlinear electrodynamics described by the Weisskopf effective Lagrangian. In particular, if one chooses f in the form

$$f(\mathcal{F},\mathcal{G})=\left(\frac{e_0}{r_0}\right)^2\left\{1+\frac{1}{8}\pi^{-2}\mathcal{F}^{-1}\int_0^\infty s^{-3}\exp(-m^2s)\times\left[(es)^2\mathcal{G}\frac{\mathrm{Re}\cosh(es\mathcal{X})}{\mathrm{Im}\cosh(e_0s\mathcal{X})}-1-\frac{2}{3}(es)^2\mathcal{F}\right]ds\right\}^{-2/3} \tag{3.1}$$

then the Kaluza-Klein metric

$$d\mathcal{S}^2=\eta_{\mu\nu}dx^\mu dx^\nu-f^{-1}(d\theta-A_\nu dx^\nu)^2 \tag{3.2}$$

leads to the Weisskopf Lagrangian density

$$\mathcal{L}=-\mathcal{F}-\frac{1}{8}\pi^{-2}\int_0^\infty s^{-3}\exp(-m^2s)\times\left[(es)^2\mathcal{G}\,\frac{\mathrm{Re}\cosh(es\mathcal{X})}{\mathrm{Im}\cosh(ex\mathcal{X})}-1-\frac{1}{2}(es)^2\mathcal{F}\right]ds. \tag{3.3}$$

To obtain the weak field limit Lagrangian (1.5), one uses the Kaluza-Klein metric (3.2) with f taken to be

$$f=\left(\frac{e_0}{r_0}\right)^2\left(1+\frac{4\alpha^2}{45m^2}\frac{(\vec{E}^2-\vec{B}^2)^2+7(\vec{E}\cdot\vec{B})^2}{\vec{E}^2-\vec{B}^2}\right)^{-2/3}. \tag{3.4}$$

The nonlinear field equations then follow from the Euler-Lagrange equations in the usual way.

This shows that it is possible to give a Kaluza-Klein type formulation for the nonlinear electrodynamics of the Weisskopf effective Lagrangian. In fact, any nonlinear electrodynamics could be placed in the Kaluza-Klein frame work in this fashion. While it has been known for some time that every theory which is generally covariant and gauge invariant can be expressed in Kaluza-Klein form [11,12], this does not seem to have ever been explicitly done for any cases other than linear electrodynamics and its non-Abelian gauge theory extensions. The results of this section should also be of interest in their own right because they show the effect of quantum corrections on the five-dimensional metric in a simplified Kaluza-Klein model.

IV. MOTION OF PARTICLES IN NONLINEAR ELECTRODYNAMICS DESCRIBED BY AN EFFECTIVE LAGRANGIAN

It is possible to use the Kaluza-Klein formalism to derive the equation of motion for a test particle in the nonlinear electromagnetic field described by the Weisskopf Lagrangian. The method of calculation is straightforward since the motion of test particles in Kaluza-Klein theory is along geodesics of the five-dimensional metric, which extremize the arc length. For a particle of rest mass m_0 this is equivalent to finding the equations of motion that follow from the particle action

$$\mathcal{A}_p=-m_0\int d\mathcal{S}. \tag{4.1}$$

Using (3.2) this can be written as

$$\begin{aligned}\mathcal{A}_p&=-m_0\int (\eta_{\mu\nu}dx^\mu dx^\nu-f^{-1}(d\theta-A_\nu dx^\nu)^2)^{1/2}\\&=-m_0\int \sqrt{1-\vec{v}^2-f^{-1}(\dot{\theta}-A_0+\vec{A}\cdot\vec{v})^2}dt,\end{aligned} \tag{4.2}$$

where f is given by (3.3) and $\vec{v}=d\vec{x}/dt$ is the usual velocity vector. The time rate of change of the fifth component of the particle's position is $\dot{\theta}=d\theta/dt$.

The particle motion is thus described by the Lagrangian

$$L=-m_0\sqrt{1-\vec{v}^2-\frac{1}{f}(\dot{\theta}-A_0+\vec{A}\cdot\vec{v})^2}. \tag{4.3}$$

The canonical momenta resulting from (4.3) are then

$$Q\equiv\frac{\partial L}{\partial\dot{\theta}}=\frac{m_0f^{-1}(\dot{\theta}-A_0+\vec{A}\cdot\vec{v})}{\sqrt{1-\vec{v}^2-\frac{1}{f}(\dot{\theta}-A_0+\vec{A}\cdot\vec{v})^2}}. \tag{4.4}$$

and

$$\vec{P}\equiv\frac{\partial L}{\partial\vec{v}}=\frac{m_0\vec{v}}{\sqrt{1-v^2-f^{-1}(\dot{\theta}-A_0+\vec{A}\cdot\vec{v})^2}}+Q\vec{A}. \tag{4.5}$$

So that Lagrange's equations lead to the equations of motion

$$\frac{dQ}{dt}=0, \tag{4.6a}$$

$$\begin{aligned}\frac{d\vec{P}}{dt}&=Q\vec{\nabla}(-A_0+\vec{A}\cdot\vec{v})\\&-\frac{Q^2}{2m_0}\sqrt{1-\vec{v}^2-f^{-1}(\dot{\theta}-A_0+\vec{A}\cdot\vec{v})^2}\ \vec{\nabla}f.\end{aligned} \tag{4.6b}$$

The first of these shows that $Q\equiv e=\text{const}$ which in Kaluza-Klein theory is taken to be the observed charge of the test particle. Using (4.5) the second equation becomes

$$\begin{aligned}&\frac{d}{dt}\left(\frac{m_0\vec{v}}{\sqrt{1-\vec{v}^2-\frac{1}{f}(\dot{\theta}-A_0+\vec{A}\cdot\vec{v})^2}}\right)\\&=e(\vec{E}+\vec{v}\times\vec{B})\\&-\frac{e^2}{2m_0}\sqrt{1-\vec{v}^2-\frac{1}{f}(\dot{\theta}-A_0+\vec{A}\cdot\vec{v})^2}\ \vec{\nabla}f.\end{aligned} \tag{4.7}$$

From (4.4) it can be shown that

$$\frac{1}{\sqrt{1-\vec{v}^2-\frac{1}{f}(\dot{\theta}-A_0+\vec{A}\cdot\vec{v})^2}}=\left(1+\frac{e^2f}{m_0}\right)^{1/2}\gamma \tag{4.8}$$

with $\gamma=(1-\vec{v}^2)^{-1/2}$, so that (4.7) can be written as

$$\frac{d}{dt}(m\gamma\vec{v})=e(\vec{E}+\vec{v}\times\vec{B})-\frac{e^2}{2m\gamma}\vec{\nabla}f, \tag{4.9}$$

where

$$m\equiv\sqrt{m_0^2+e^2f} \tag{4.10}$$

is interpreted in the Kaluza-Klein theory as the renormalized mass of the test particle.

From (4.9) one can easily obtain the covariant equation of motion

$$\frac{d}{d\tau}(mu^\mu)=eF^{\mu\nu}u_\nu+\frac{e^2}{2m}f^{,\mu}, \tag{4.11}$$

which upon using (4.10) can be written as

$$\frac{d}{d\tau}(mu^\mu)=eF^{\mu\nu}u_\nu+m^{,\mu} \tag{4.12}$$

where τ is the usual proper time and $u^\mu=dx^\mu/d\tau$ is the usual four velocity. It is important to note that the derivation of these equations of motion is applicable in arbitrary fields and, in particular, does not rely on a weak field or slowly varying field approximation.

For the case of weak fields f is given by (3.4) and in this case the equation of motion (4.12) becomes

$$\frac{d}{dt}(m\vec{v})=e(\vec{E}+\vec{v}\times\vec{B})+\frac{8\alpha^2}{135m^3}\vec{\nabla}\left[\frac{(\vec{E}^2-\vec{B}^2)^2+7(\vec{E}\cdot\vec{B})^2}{2(\vec{E}^2-\vec{B}^2)}\right] \tag{4.13}$$

for slowly moving particles. It is informative to consider (4.13) for the case where $\vec{E}\cdot\vec{B}$ vanishes so that (4.13) reduces to

$$\frac{d}{dt}(m\vec{v})=e(\vec{E}+\vec{v}\times\vec{B})+\frac{8\alpha^2}{135}\left(\frac{\hbar}{mc}\right)^3\vec{\nabla}\left(\frac{\varepsilon_0\vec{E}^2-\varepsilon_0c^2\vec{B}^2}{2}\right), \tag{4.14}$$

where we have restored the constants $\hbar$ and c and expressed the fields in rationalized mks units.

The equation of motion (4.14) shows the quantum corrections to the motion of a classical test particle. It is obvious that the first term corresponds to the Lorentz force and the second arises due to nonlinear quantum effects resulting from vacuum polarization. The effect of the additional term is a weak diffusion-type force on the particle pulling it into regions with high energy density. Because of its presence, the motion of charged particles will be correspondingly modified in intense electromagnetic fields.

To obtain an estimate of the magnitude of the additional term, we compare it to the Lorentz force for an electron in a hydrogen atom ground state orbital. Neglecting the magnetic field, the electric field of the proton provides the external field and the ratio of the diffusive force on the orbital electron to the Coulomb force of the nucleus on the orbital electron is

$$\frac{F_d}{F_e}=\frac{\frac{8\alpha^2}{135}\left(\frac{\hbar}{mc}\right)^3\varepsilon_0 E\frac{\partial E}{\partial r}}{eE}=\frac{8\alpha^2}{135}\left(\frac{\hbar}{mc}\right)^3\frac{\varepsilon_0}{e}\frac{\partial E}{\partial r}, \tag{4.15}$$

which for the Coulomb field $E=\frac{1}{4\pi\varepsilon_0}\frac{e}{r^2}$ yields

$$\frac{F_d}{F_e}=\frac{8\alpha^2}{270\pi}\left(\frac{\hbar}{mc}\right)^3\frac{1}{a_0^3}\approx 10^{-13} \tag{4.16}$$

at $r=a_0$, the first Bohr radius.

V. CONCLUSIONS

The principal result of this paper is the derivation of the equation of motion for a test particle in the nonlinear electromagnetic field described by the Weisskopf Lagrangian. Vacuum polarization effects are seen to lead to quantum corrections in the Lorentz force law. The method used to obtain the equation of motion is somewhat unusual in that an extra dimension is introduced and an equivalent Kaluza-Klein formulation is given for the electrodynamics described by the Weisskopf Lagrangian. The equation of motion is then obtained as the geodesic equation in this higher dimensional space time. This Kaluza-Klein formulation should be of interest in itself because it shows how to incorporate quantum corrections into the Kaluza-Klein scheme. The theory reduces to the original Kaluza-Klein formalism for classical electrodynamics in the limit $f\rightarrow 1$.

The technique of introducing an extra coordinate is not specific to the system studied here and could be used to find the equation of motion for test particles in any gauge invariant nonlinear electrodynamics. It should also be possible to use the same technique in obtaining quantum corrections to non-Abelian gauge theories by introducing more than five dimensions. An interesting extension of the results here would be to find the effective equation of motion for a Dirac particle in the nonlinear electromagnetic field described by the Weisskopf Lagrangian. This could be done by formulating a Dirac equation for the space described by the five-dimensional metric (3.2). The resulting quantum mechanical theory should give some insight into vacuum polarization effects on the evolution of a quantum system. Further investigations along these lines are currently under way.

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